Math 249 Wednesday, April 22

$H_\lambda(x;q)$ $\qquad K_{\lambda\mu}(q) = \langle s_\lambda, H_\mu(x;q)\rangle$

$K_{\lambda\mu}(1) = K_{\lambda\mu} = |SSYT(\lambda,\mu)|$ $\qquad H_\mu(x;1) = h_\mu$

$$K_{\lambda\mu}(q) = \sum_{T\in SSYT(\lambda,\mu)} q^{c(T)} \qquad c(T) = \underline{\text{charge}} \text{ of } T$$

$$\begin{array}{llll} 4 & 5 & & \\ 2 & 3 & 3 & \\ 1 & 1 & 1 & 2 \end{array} \qquad 4_1 5_2 2_0 3_1 3_1 1_0 1_0 1_0 2_1$$

$c(T) = 6$ $\qquad 1+2+1+1+1 = 6$

$c(T)$ is jeu-de-taquin invariant

$$C(xW) = 1 + c(Wx) \qquad x \neq 1$$

$$\boxed{1_0 1_0 1_0 1_0 2_1 2_1 2_1 3_2 3_2 4_3} \qquad c = \sum_i (i-1)\mu_i = n(\mu)$$

$$\text{Co-}\underline{\text{charge}} \quad cc(T) = n(\mu) - c(T)$$

$$\begin{array}{llll} 3 & & & \\ 2 & 2 & & \\ 1 & 1 & 1 & \\ 0 & 0 & 0 & 0 \end{array} = 10$$

Ex. $\begin{array}{l} 4 \\ 3 \\ 2 \\ 1 \end{array}$ $\quad c=0 \quad cc = \binom{n}{2}$ $\qquad 1\,1\,1\,1\,2\,2\,3\,4 \quad c = n(\mu) \quad cc = 0$

$$cc(xW) = cc(Wx) - 1$$

To compute $cc(T)$: cyclage

$cc(T) = 7$

$$\begin{array}{llll} 4 & 5 & & \\ 2 & 3 & 3 & \\ 1 & 1 & 1 & 2 \end{array} \xrightarrow{1} \begin{array}{llll} 5 & & & \\ 2 & 3 & 3 & \\ 1 & 1 & 1 & 2 \end{array} \; 4 \xrightarrow{\text{jdt}} \begin{array}{lllll} 5 & & & & \\ 2 & 3 & 3 & & \\ 1 & 1 & 1 & 2 & 4 \end{array} \quad 2$$

$$\begin{array}{lllll} 2 & 3 & 3 & & \\ 1 & 1 & 2 & 4 & 5 \end{array} \xrightarrow[3]{} \begin{array}{lllll} 3 & 3 & 4 & & \\ 1 & 1 & 2 & 2 & 5 \end{array} \xrightarrow[4]{} \begin{array}{lllll} 3 & 4 & 5 & & \\ 1 & 1 & 2 & 2 & 3 \end{array} \xrightarrow[5]{} \begin{array}{llllll} 4 & 5 & & & & \\ 1 & 1 & 2 & 2 & 3 & 3 \end{array} \xrightarrow[6]{} \begin{array}{lllllll} 5 & & & & & & \\ 1 & 1 & 2 & 2 & 3 & 3 & 4 \end{array} \xrightarrow[7]{} 1\,1\,2\,2\,3\,3\,4\,5$$

$$\mu = 3\,2\,2\,1\,1 \qquad n(\mu) = 13$$

$$cc(T) = 13 - c(T) = 13 - 6 = 7. \qquad \begin{array}{lll} 4 & & \\ 3 & & \\ 2 & 2 & \\ 1 & 1 & \\ 0 & 0 & 0 \end{array}$$

$$\begin{array}{l} 45 \\ 233 \\ \hline 1112 \end{array} \qquad 1112 \xrightarrow{5} \qquad \begin{array}{l} 45 \\ 233 \end{array} \qquad \begin{array}{l} \boxed{45} \quad 2 \\ 111\,2233 \end{array}$$

$$111\,2233 \qquad 45 \qquad 10 \qquad 111\,22\,33\,45$$

$$cc(T) = 5 + 2 = 7$$

Ex.

$$\begin{array}{l} 7 \\ 356 \\ 124 \end{array} \qquad 7_3\, 3_1\, 5_2\, 6_3\, 1_0\, 2_1\, 4_2$$

$3 \notin D(T)$

$1 \notin D(T)$

$$T \in SYT \Rightarrow c(T) = \sum_{i \notin D(T)} n - i \qquad \mu = (1^n) \qquad n(\mu) = \binom{n}{2}$$

$$cc(T) = \sum_{i \in D(T)} n-i = \text{(reverse maj)}$$

$$K_{\lambda,(1^n)}(q) = \sum_{T \in SYT(\lambda)} q^{cc(T)} = \sum_{T \in SYT(\lambda)} q^{maj(T)}$$

$$maj(T) = \sum_{i \in D(T)} i \qquad cc(T) = maj(ev(T))$$

$$\pi \mapsto w_0 \pi w_0 \qquad P(\pi), Q(\pi) \mapsto ev(P), ev(Q)$$

Rep Theory / Geometry point of view

$$S_\lambda = \sum_{w \in S_n} w \left(\frac{x^\lambda}{\prod_{i<j} (1 - x_j/x_i)} \right)$$

Irr. Char. of GL_n

$$\prod_{\alpha \in R+} (1 - x^{-\alpha}) \qquad \alpha = e_i - e_j \quad i<j$$

x_i/x_j

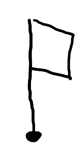
$$\begin{pmatrix} 1 & x & x & x & x & x \\ x & 1 & x & x & x & x \\ & & 1 & x & x & x \\ & & & 1 & x & x \\ & & & & \ddots & \\ & & & & & 1 \end{pmatrix} \qquad \begin{pmatrix} x_1 & & \\ & \ddots & \\ & & x_n \end{pmatrix}$$

$G = GL_n(\mathbb{C})$ B = upper triangular subgroup (Borel)

$G = SO_n$, Sp_{2n}, exceptionals ...

Flag variety $X = G/B$

$$GL_n: \quad X = \{ 0 \subset F_1 \subset F_2 \subset \cdots \subset F_{n-1} \subseteq \mathbb{C}^n \mid \dim F_i = i \}$$



$$E \qquad E_i = \langle e_1, \dots, e_i \rangle$$

$$B = \mathrm{Stab}(E_\cdot)$$

$$B \begin{pmatrix} a_1 & & * \\ & \ddots & \\ 0 & & a_n \end{pmatrix} \longmapsto \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix}$$

$B \to T$ = diagonal matrices

$$\begin{pmatrix} x \\ x \\ x \\ \vdots \\ 0 \\ 0 \end{pmatrix}$$

$$B \to T \qquad B \curvearrowright \mathbb{C}$$

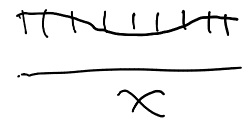
$$T \curvearrowright \mathbb{C} \qquad a_1 \cdots a_n \curvearrowright a_1^{\lambda_1} \cdots a_n^{\lambda_n}$$

$$\mathbb{C}_\lambda \qquad G \times_B \mathbb{C} \qquad (g, v) \Big/ (gb, v) \approx (g, bv)$$

$$b \cdot (g, v) = (gb^{-1}, bv) \qquad (G \times \mathbb{C})/B$$

$(g,w) \mapsto gB \in G/B$

$G \times_B \mathbb{C}_\lambda \to X = G/B$ 1-dimensional $\mathbb{C}$ vector bundle (line bundle).



$\mathcal{L}_\lambda = G \times_B \mathbb{C}_{w_0(\lambda)}$

$H^0(X, \mathcal{L}_\lambda)$ = The irr. rep. of G with character $\sum_{w\in W} w\left(\frac{x^\lambda}{\prod_{\alpha\in R_+}(1-x^{-\alpha})}\right)$

T fixed points on G/B are wB : $w \in W$ $\quad \uparrow S_n = N(T)/T = S_n$

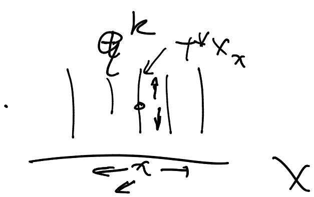
$\sum (-1)^i \chi_{H^i(X,\mathcal{L}_\lambda)}$ = Atiyah-Bott-Lefschetz formula in terms of $(G/B)^T$

$H^i(X,\mathcal{L}_\lambda) = 0$ for $i \neq 0$

$\sum_{w\in W} w\left(\frac{x^\lambda}{\prod_{\alpha\in R_+}((1-x^{-\alpha})(1-qx^{\alpha}))}\right)$ $\quad T\times\mathbb{C}^*$ $\quad G \curvearrowright T^*X$

$\times$ $\mathbb{C}^*$ $\leftarrow$ scalar mult. q

$\uparrow$ weights on T^*X at $1B$ or E.



G/B

gives $G \times \mathbb{C}^*$ character of T^*X

Any G

$H^i(T^*X, \mathcal{L}_\lambda) = 0$ for $i>0$ $\quad H^0(T^*X, \mathcal{L}_\lambda)$

$K_{\lambda\mu}(q) = $ multiplicity of χ_λ (irr. of G) in $H^0(T^*X, \mathcal{L}_\mu) \Leftarrow$

$K_{\lambda\mu}(q) = \sum_{w\in W}(-1)^{\ell(w)} P_q(w(\lambda+\rho)-\mu-\rho)$ $\quad \Rightarrow K_{\lambda\mu}(q) \in \mathbb{N}[q]$

(Another, different, geometric reason too)

$K_{\lambda\mu}(1) = K_{\lambda\mu}$ = multiplicity of weight μ in χ_λ.

Back to GL_n, $H_\lambda(x;q)$, $K_{\lambda\mu}(q)$ $\quad m \geq \lambda_1$ or $m < \lambda_1$

Jing Operators $\quad$ Recall Bernstein Ops $B_m S_\lambda = S_{(m,\lambda)}$ $\quad S_{-1\,3} = -S_2$

$B_m = \langle z^m \rangle\, \Omega[zX]\,\Omega[-z^{-1}X]^{\perp}$

$$\begin{array}{rr} (-1\ \ 3) & (3\ \ 0) \\ +(1\ \ 0) & -(1\ \ 0) \\ (0\ \ 3) & (2\ \ 0) \end{array}$$

$$\mathbb{H}_m^q = \langle z^m \rangle\, \Omega[zX]\, \Omega[(q-1) z^{-1} X]^{\perp}$$

$$\Omega[(q-1)z^{-1}X]^{\perp} = \Omega[qz^{-1}X]^{\perp}\Omega[-z^{-1}X]^{\perp}$$

Prop. $\mathbb{H}_m^q H_\lambda(x;q) = H_{(m;\lambda)}(x;q)$

(If $m \geq \lambda_1$, or $m < \lambda_1$)

$$= \langle z^m \rangle\, \Omega[zx]\, \Omega[-z^{-1}X]^{\perp}\, \Omega[qz^{-1}X]^{\perp}$$

z^{m+k} $\quad$ z^{-k} $\quad$ $\Omega[qz^{-1}X] = \sum q^k z^{-k} h_k(x)$

$$= \sum_{k=0}^{\infty} q^k B_{m+k}\, h_k(x)^{\perp}$$

$$\mathbb{H}_m^q S_\lambda = \sum_k q^k \sum_{\substack{|\nu| = |\lambda| - k \\ \lambda/\nu \text{ Hstrip}}} S_{(m+k;\nu)}$$

$$\sum_w w\left(\frac{x_1^m\, x_2^{\lambda_1} \cdots x_\ell^{\lambda_{\ell-1}}}{\prod_{i<j} (1 - x_j/x_i)} \right) = S_{(m;\lambda)}$$

$$f(x)^{\perp} s_\lambda = \sum w\left(\frac{x^\lambda\, f(x_1^{-1}, \dots, x_\ell^{-1})}{\prod (1 - x_j/x_i)} \right)$$

$$\sum_{\substack{k,\ \nu_i \leq \lambda_i \\ |\nu| = |\lambda| - k}} q^k w \left(\frac{x_1^{m+k} x_2^{\nu_1} \cdots x_\ell^{\nu_{\ell-1}}}{\prod_{i<j} (1 - x_j/x_i)} \right)$$

$(x_1/x_j)^{\cdots}$ $\quad j \neq 1$

$$= \sum_w w \left(\frac{x_1^m\, x_2^{\lambda_1} \cdots x_\ell^{\lambda_{\ell-1}}}{\prod_{i<j} (1 - x_j/x_i) \prod_{j \neq 1} (1 - q x_1/x_j)} \right)$$

H_λ

x_1/x_j

$$\frac{x_1^m\, s_\lambda(x_2, \dots, x_{\ell-1})}{\prod_{j \neq 1} ((1 - x_j/x_1)(1 - q x_1/x_j))}$$

$$\mathbb{H}_m^q f = \sum_{w \in S_n/S_1 \times S_{n-1}} w \left(\frac{x_1^m\, f(x_2, \dots, x_{\ell-1})}{\prod_{j \neq 1} ((1 - x_j/x_1)(1 - q x_1/x_j))} \right)$$

$$S_{m;\lambda} = \sum_{w \in S_n/S_1 \times S_{n-1}} w \left(\frac{x_1^m\, s_\lambda(x_2, \dots, x_{\ell-1})}{\prod_{j \neq 1} (1 - x_j/x_1)} \right)$$

$$\mathbb{H}_m^q H_\lambda(x;q) = H_{m;\lambda}(x;q)$$

$$H_\lambda(x;1) = h_\lambda \Leftarrow K_{\lambda\mu}(1) = K_{\lambda\mu}$$

$$H_\lambda(x;q) = \mathbb{H}_{\lambda_1}^q \cdots \mathbb{H}_{\lambda_k}^q \cdot 1$$

$$H_\lambda(x;0) = B_{\lambda_1} \cdots B_{\lambda_k} \cdot 1$$

$$= S_\lambda$$

$$H_\lambda(x;1) = h_\lambda$$

$$\mathbb{H}_m^q = \langle z^m \rangle\, \Omega[zX]\, \Omega[(q-1) z^{-1} X]^{\perp}$$

$$\mathbb{H}_m^0 = B_m$$

$$\mathbb{H}_m^1 = \langle z^m \rangle\, \Omega[zX]$$

$$= h_m \text{ (i.e. multiplication by } h_m)$$

$$\Omega[zX] = \sum z^m h_m(x)$$

Upper + Lower Triangularity & Orthogonality

<u>Upper</u>: $K_{\lambda\mu}(q) = 0$ unless $\lambda \geq \mu$ $\quad K_{\mu\mu}(q) = 1$

$$H_{\mu}(x;q) = s_{\mu} + \sum_{\lambda > \mu} K_{\lambda\mu}(q)\, s_{\lambda} = \sum_{\lambda} K_{\lambda\mu}(q)\, s_{\lambda}$$

<u>Lower</u>: $H_{\mu}[X(1-q);q] = \sum_{\lambda \leq \mu} b_{\lambda\mu}\, s_{\lambda}$

$\leftarrow$ Characterize H_{μ}.

Orthogonality: $\langle f, g \rangle_q \underset{def}{=} \langle f, g[X(1-q)] \rangle$

Then $\langle H_{\lambda}, H_{\mu} \rangle_q = 0$ if $\lambda \neq \mu$

Next time.